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A NEW SUBDIVISION FOR COMPUTING FIXED POINTS WITH A HOMOTOPY ALGORITHM

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In this paper a triangulation is introduced for homotopy methods to compute fixed points on the unit simplex or in R^n. This triangulation allows for factors of incrementation of more than two. The factor may be of any size and even different at each level. Also the starting point on a new level may be any gridpoint of the last found completely labelled subsimplex on the last level. So, the decision which new factor of incrementation and which starting point is used, can be made on the ground of previous approximations. Doing so, the convergence rate can be accelerated without using restart methods.

Key words: Triangulation; Homotopy Function; Fixed Point; Grid Refinement.

1. Introduction

To compute a fixed point of an upper semi-continuous point-to-set mapping ψ, Eaves [1] and Eaves and Saigal [2] introduced a homotopy algorithm using a triangulation of $S^n \times [1, \infty)$ respectively $R^n \times [0, \infty)$ with a continuous refinement of the grid size. In their algorithms the mapping ψ is deformed from a linear function ψ^0 on level f_0 to a piecewise linear approximation ψ^k on level f_k. For k goes to infinity ψ^k converges to the mapping ψ. The proposed triangulation of $S^n \times [1, \infty)$ or $R^n \times [0, \infty)$ is built up from triangulations between two successive levels. As discussed by Todd [14] only triangulations with factors of incrementation of at most two are known. However, in restart algorithms (see e.g. Merrill [10], Kuhn and MacKinnon [4], Van der Laan and Talman [5, 6, 8], and Reiser [11]) any factor of incrementation can be used. In this paper we introduce a triangulation of $S^n \times [1, \infty)$ and $R^n \times [0, \infty)$, such that between any two successive levels the factor of incrementation can be of any size. The algorithm generates from $S^n = \{1\}$ a path of adjacent simplices. Let us assume that σ_m is the first simplex of the triangulation on level f_m generated by the algorithm. Then the factor of incrementation k_m to obtain the new level f_{m+1} can be chosen arbitrarily. Moreover, the gridpoint on level f_{m+1} connected with $\sigma_m \times \{f_m\}$ can be any gridpoint $v(\sigma_m)$ of $\sigma_m \times \{f_{m+1}\}$. The choice of k_m and $v(\sigma_m)$ can be of course on ground of the information obtained from the subsimplex $\sigma_m \times \{f_m\}$. As soon as k_m and $v(\sigma_m)$ are chosen, the triangulation of $S^n \times [f_m, f_{m+1}]$ or $R^n \times [f_m, f_{m+1}]$ is fixed throughout the rest of the algorithm.

In Section 2 we describe the triangulation for the unit simplex S^n. In Section 3 we give a concise description of the algorithm, whereas the replacement steps are described in Section 4. In Section 5 we discuss the triangulation for R^n. Finally, in Section 6 some concluding remarks are made.

2. Triangulation of $S^n \times [1, \infty)$

Let S^n be the n-dimensional unit simplex, i.e.

$$S^n = \{x \in R_+^{n+1} \,|\, \textstyle\sum_{i=1}^{n+1} x_i = 1\}.$$

The vertices of S^n are $e(i)$, $i = 1, \dots, n+1$, where $e(i)$ is the ith $(n+1)$-dimensional unit vector. To triangulate $S^n \times [1, \infty)$ we choose an arbitrary sequence of increasing integers $f_0, f_1, \dots$ such that f_{m+1} is a multiple k_m of f_m and $f_0 = 1$. We will describe only the triangulation of $S^n \times [f_m, f_{m+1}]$. Combining then the triangulations of $S^n \times [f_i, f_{i+1}]$ for all pairs (f_i, f_{i+1}) we obtain directly the triangulation of $S^n \times [1, \infty)$. Let for $h = 0, 1, \dots$, G_h be the standard triangulation of S^n with grid size f_h (see Kuhn [3]), i.e. G_h is the collection of n-simplices $\tau(v^1, \gamma)$ with vertices $v^1, \dots, v^{n+1}$ of S^n such that

(i) $\gamma = (\gamma_1, \dots, \gamma_{n+1})$ is a permutation of the elements of $I_{n+1} = \{1, \dots, n+1\}$;
(ii) the components of v^1 are a multiple of f_h^{-1};
(iii) $v^{j+1} = v^j + q(\gamma_j)/f_h$, $j = 1, \dots, n$

where $q(j)$ is the jth column of the $(n+1) \times (n+1)$-matrix Q defined by

$$Q = \begin{bmatrix} -1 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ 1 & -1 & & & & & 0 \\ 0 & 1 & \cdot & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & -1 & 0 \\ 0 & \cdot & & \cdot & 0 & 1 & -1 \end{bmatrix}.$$

Observe that each component of v^j, $j = 1, \dots, n+1$, is a multiple of f_h^{-1} and that $v^1 = v^{n+1} + q(\gamma_{n+1})/f_h$. Since any vertex of a simplex can be chosen to be v^1, each simplex has $n+1$ representations. However, in the following, it will be more appropriate to represent an n-simplex of S^n in a unique way. This will be done as follows.

For a given gridpoint y let

$$\alpha_i(y) = \left(1 - \sum_{j=1}^{i} y_j\right) f_h, \quad i = 1, \dots, n.$$

Clearly, every $\alpha_i(y)$ is an integer. Now, define $\chi(y)$ by

$$\chi(y) = 1 + \left(\sum_{j=1}^{n} \alpha_j(y)\right) \mod(n+1).$$

Because of the structure of a simplex, we have that $\chi(w^i) \neq \chi(w^j)$ for every simplex $\sigma(w^1, \ldots, w^{n+1})$ if $i \neq j$. So, each simplex has a unique vertex w^{i_j} with $\chi(w^{i_j}) = j$, $j = 1, \ldots, n+1$. Then the representation $\tau(v^1, \gamma)$ is chosen such that $v^1 = w^{i_1}$. It is easy to see that $v^j = w^{i_j}$, $j = 1, \ldots, n+1$. Note that any gridpoint y has the same index $\chi(y)$ in all simplices of which it is a vertex. In the following we will assume that every n-simplex which is a simplex of the triangulation of S^n with grid size f_h, $h = 0, 1, \ldots$, is represented in this way.

To construct a triangulation of $S^n \times [f_m, f_{m+1}]$ for given m, $m = 0, 1, \ldots$, let $\sigma_m(u^1, \beta)$ be a particular simplex of G_m called the starting simplex on level f_m, represented as described in the previous paragraph. Let $u(\sigma_m)$ be a particular gridpoint of the triangulation G_{m+1} in σ_m, i.e. there are unique nonnegative integers $\lambda_1^m, \ldots, \lambda_{n+1}^m$ with sum equal to k_m such that

$$u(\sigma_m) = \sum_{i=1}^{n+1} \lambda_i^m u^i / k_m,$$

where $u^{i+1} = u^i + q(\beta_i)/f_m$, $i = 1, \ldots, n$.

In the sequel we will call the vertex

$$v(\tau) = \sum_{i=1}^{n+1} \lambda_i^m v^i / k_m$$

the centrepoint of the simplex $\tau(v^1, \gamma)$. The triangulation of $S^n \times [f_m, f_{m+1}]$ will be such that the $(n+1)$-simplex which is the convex hull of $\tau \times \{f_m\}$ and $v(\tau) \times \{f_{m+1}\}$, is a simplex of this triangulation. In particular the $(n+1)$-simplex ϕ_m, which is the convex hull of the starting simplex σ_m on level f_m and its centrepoint $u(\sigma_m)$ on level f_{m+1}, will be a simplex of this triangulation since $v(\tau) = u(\sigma_m)$ if $\tau(v^1, \gamma) = \sigma_m(u^1, \beta)$. If $m = 0$, $\sigma_m = S^n$ and G_m consits of only one simplex. To triangulate $S^n \times [f_m, f_{m+1}]$ we first triangulate $\tau(v^1, \gamma) \times [f_m, f_{m+1}]$ for an arbitrary n-simplex $\tau(v^1, \gamma)$ of G_m, represented in the way as described above. Then we will prove that the union of the triangulations over all n-simplices $\tau(v^1, \gamma)$ is a triangulation of $S^n \times [f_m, f_{m+1}]$. To triangulate the set $\tau(v^1, \gamma) \times [f_m, f_{m+1}]$, define for any proper subset T of I_{n+1} the set of gridpoints $A^\tau(T)$ of the triangulation G_{m+1} in $\tau(v^1, \gamma)$ by

$$A^\tau(T) = \{y \in \tau \mid y = v(\tau) + \textstyle\sum_{h \in T} \mu_h q(\gamma_h)/f_{m+1} \text{ for positive integers } \mu_h,\ h \in T\}.$$

Note that the grid points of τ are partitioned in this way. In particular $A^\tau(\emptyset) = v(\tau)$. A triangulation of $\tau \times [f_m, f_{m+1}]$ is obtained if all gridpoints x in $A^\tau(T)$ on level f_{m+1} are connected with the vertices v^i on level f_m, $i \notin T$ (cf. Van der Laan and Talman [5] and Todd [14]). So, $v(\tau)$ on level f_{m+1} is connected with all the vertices of τ on level f_m.

Theorem 2.1. *The union of the triangulations of $\tau(v^1, \gamma) \times [f_m, f_{m+1}]$ over all simplices $\tau(v^1, \gamma)$ of G_m triangulates $S^n \times [f_m, f_{m+1}]$.*

Before proving this theorem, we will illustrate it in Fig. 1 for $n = 2$; $f_1 = 4$; $f_2 = 16$; $\lambda^0 = (1, 1, 2)$; $\lambda^1 = (2, 1, 1)$. In Fig. 1a the triangulation G_1 of S^n is given. The centre point of $\sigma_0 = S^2$ is $u = u(\sigma_0) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. The gridpoints of G_1 in the region $A(T)$ are connected with $e(i)$, $i \notin T$. Let $\sigma_1(u^1, B)$ where $u^1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and $\beta = (1, 3, 2)$ be the starting simplex of G_1. Then

$$u(\sigma_1) = \sum_{i=1}^{n+1} \lambda_i^1 u^i = (\tfrac{7}{16}, \tfrac{3}{8}, \tfrac{3}{16}).$$

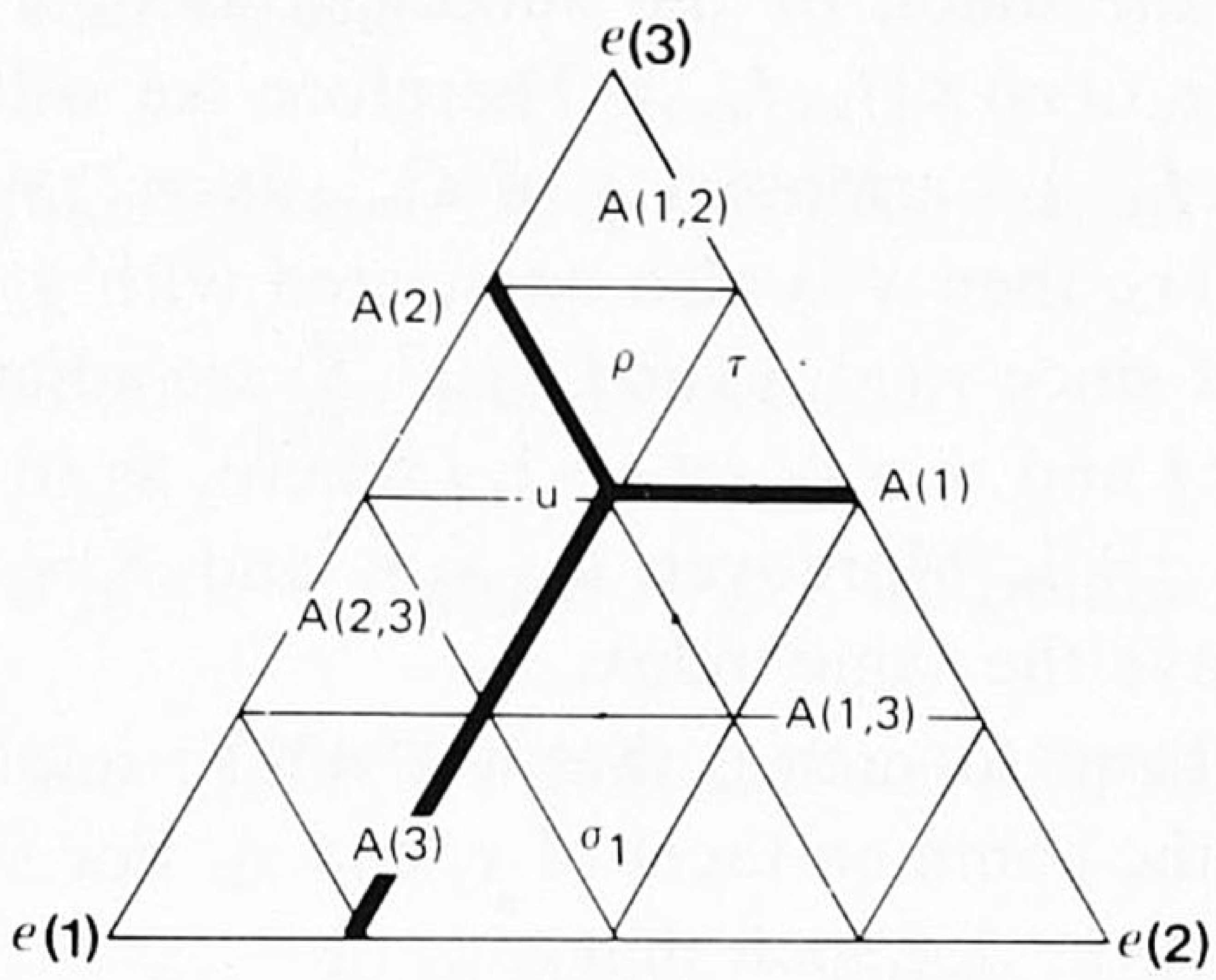

Fig. 1a. $n = 2$, $f_1 = 4$, $\lambda^0 = (1, 1, 2)$, $u(\sigma_0) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$.

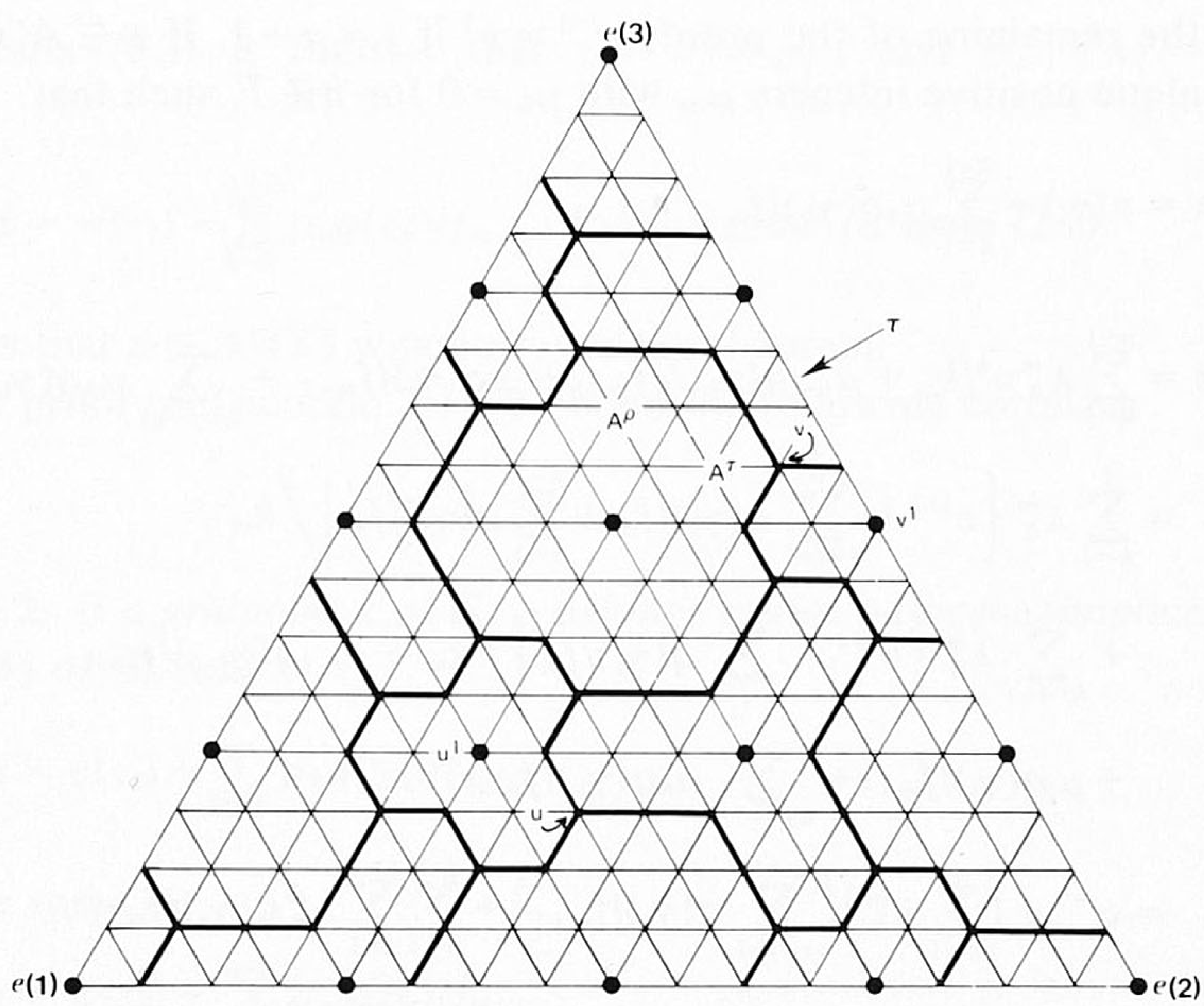

Fig. 1b. $n = 2$, $f_2 = 16$, $\lambda^1 = (2, 1, 1)$, $\sigma_1 = \sigma_1(u^1, \beta)$ with $u^1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and $\beta = (1, 3, 2)$, $u = u(\sigma_1)$, $v = v(\tau)$ where $\tau = \tau(v^1, \gamma)$ with $v^1 = (0, \frac{1}{2}, \frac{1}{2})$ and $\gamma = (2, 3, 1)$, $A^\tau = A^\tau(1, 2)$ and $A^\rho = A^\rho(1, 2)$. The vertices of G_1 are drawn heavily. The gridpoints of G_2 within or on the boundary of a region surrounded by heavy lines are connected with the point of G_1 in the middle.

Let $\tau=\tau(v^1,\gamma)$ be the simplex with $v^1=(0,\frac{1}{2},\frac{1}{2})$ and $\gamma=(2,3,1)$. So $v(\tau)=(\frac{1}{16},\frac{3}{8},\frac{9}{16})$. In Fig. 1b the triangulation G_2 is pictured. The gridpoints of G_2 in the regions $A^\tau(T)$ are connected with the vertices v^i, $i\notin T$, of τ. Observe that in the adjacent simplex ρ the region $A^\rho=A^\rho(1,2)$ is adjoining to the region $A^\tau=A^\tau(1,2)$ on a consistent way since the vertex $v^3=(\frac{1}{4},\frac{1}{4},\frac{1}{2})$ is a common vertex of τ and ρ.

Proof of Theorem 2.1. It is sufficient to show that if two simplices of G_m, say τ_1 and τ_2, are adjacent, the union of the subdivisions of $\tau_1\times[f_m,f_{m+1}]$ and $\tau_2\times[f_m,f_{m+1}]$ triangulates $(\tau_1\cup\tau_2)\times[f_m,f_{m+1}]$. Therefore we will prove that if in the subdivision of $\tau_1\times[f_m,f_{m+1}]$ a gridpoint x of G_{m+1} in $\tau_1\cap\tau_2$ is connected with a vertex y of G_m in $\tau_1\cap\tau_2$, then x is also connected with y in the subdivision of $\tau_2\times[f_m,f_{m+1}]$. Note that since $\tau_1(v^1,\gamma)$ and $\tau_2(w^1,\delta)$ are adjacent there is a unique j such that $v^i=w^i$, $i\neq j$ and $\gamma_i=\delta_i$, $i\neq j-1,j$ where, as in the remaining of the proof, $j-1=n+1$ if $j=1$. Moreover $\delta_{j-1}=\gamma_j$ and $\delta_j=\gamma_{j-1}$. So all common vertices of τ_1 and τ_2 have the same index.

Therefore we only have to prove, that $x\in A^{\tau_1}(T)$ implies $x\in A^{\tau_2}(T)$ for a gridpoint x of G_{m+1} in the common facet of τ_1 and τ_2. For such a gridpoint there are unique integers θ_i, $i\neq j-1,j$ such that

$$x=v^{j+1}+\sum_{i\neq j-1,j}\theta_i q(\gamma_i)/f_{m+1} \tag{2.1}$$

with, as in the remaining of the proof, $v^{j+1}=v^1$ if $j=n+1$. If $x\in A^{\tau_1}(T)$ there also exist unique positive integers μ_h, with $\mu_h=0$ for $h\notin T$, such that

$$x=v(\tau_1)+\sum_{h=1}^{n+1}\mu_h q(\gamma_h)/f_{m+1} \tag{2.2}$$

or

$$\begin{aligned}
x&=\sum_{h=1}^{n+1}\lambda_h^m v^h/k_m+\mu_{j-1}q(\gamma_{j-1})/f_{m+1}+\mu_j q(\gamma_j)/f_{m+1}+\sum_{h\neq j-1,j}\mu_h q(\gamma_h)/f_{m+1}\\
&=\sum_{h=1}^{j}\lambda_h^m\left\{v^{j+1}+\sum_{s=j+1}^{n+1}q(\gamma_s)/f_m+\sum_{s=1}^{h-1}q(\gamma_s)/f_m\right\}\Big/k_m\\
&\quad+\sum_{h=j+1}^{n+1}\lambda_h^m\left\{v^{j+1}+\sum_{s=j+1}^{h-1}q(\gamma_s)/f_m\right\}\Big/k_m+\mu_{j-1}q(\gamma_{j-1})/f_{m+1}\\
&\quad+\mu_j q(\gamma_j)/f_{m+1}+\sum_{h\neq j-1,j}\mu_h q(\gamma_h)/f_{m+1}\\
&=v^{j+1}+\left(\sum_{h=1}^{j}\lambda_h^m\right)\left(\sum_{s=j+1}^{n+1}q(\gamma_s)/f_{m+1}\right)+\sum_{h=1}^{j-1}\sum_{s=1}^{h-1}\lambda_h^m q(\gamma_s)/f_{m+1}\\
&\quad+\lambda_j^m q(\gamma_{j-1})/f_{m+1}+\sum_{s=1}^{j-2}\lambda_j^m q(\gamma_s)/f_{m+1}+\sum_{h=j+1}^{n+1}\sum_{s=j+1}^{h-1}\lambda_h^m q(\gamma_s)/f_{m+1}\\
&\quad+\mu_{j-1}q(\gamma_{j-1})/f_{m+1}+\mu_j\left(-\sum_{s\neq j}q(\gamma_s)\right)\Big/f_{m+1}+\sum_{h\neq j-1,j}\mu_h q(\gamma_h)/f_{m+1}.
\end{aligned}$$

Taking together the coefficients of $q(\gamma_h)$, $h \neq j-1, j$, we obtain

$$x = v^{j+1} + \sum_{h \neq j-1, j} \theta'_h q(\gamma_h)/f_{m+1} + (\lambda_j^m + \mu_{j-1} - \mu_j) q(\gamma_{j-1})/f_{m+1}$$

for unique integers θ'_h, $h \neq j-1, j$.

From formula (2.1) we obtain $\theta'_h = \theta_h$, $h \neq j-1, j$, and

$$\lambda_j^m + \mu_{j-1} - \mu_j = 0. \tag{2.3}$$

Moreover from (2.2) it follows that

$$x = \sum_{h \neq j} \lambda_h^m v^h / k_m + \lambda_j^m (v^{j-1} + q(\gamma_{j-1})/f_m)/k_m + \mu_{j-1} q(\gamma_{j-1})/f_{m+1}$$
$$+ \mu_j q(\gamma_j)/f_{m+1} + \sum_{h \neq j-1, j} \mu_h q(\gamma_h)/f_{m+1}.$$

Hence, since $w^i = v^i$, $i \neq j$, $\delta_{j-1} = \gamma_j$ and $\delta_j = \gamma_{j-1}$, we obtain

$$x = \sum_{h \neq j} \lambda_h^m w^h / k_m + \lambda_j^m (w^{j-1} + q(\delta_j)/f_m)/k_m + \lambda_j^m q(\delta_{j-1})/f_{m+1}$$
$$- \lambda_j^m q(\delta_{j-1})/f_{m+1} + \mu_{j-1} q(\delta_j)/f_{m+1} + \mu_j q(\delta_{j-1})/f_{m+1}$$
$$+ \sum_{h \neq j-1, j} \mu_h q(\delta_h)/f_{m+1}.$$

From formula (2.3) it follows that $\mu_j - \lambda_j^m = \mu_{j-1}$ and $\mu_{j-1} + \lambda_j^m = \mu_j$. Consequently,

$$x = w(\tau_2) + \sum_{h=1}^{n+1} \mu_h q(\delta_h)/f_{m+1} \quad \text{for the same } \mu_h \text{ as in (2.2).}$$

This implies that $x \in A^{\tau_2}(T)$ which proves the theorem.

From the proof of Theorem 2.1 we obtain the following corollary.

Corollary 2.2. *If a gridpoint x of G_{m+1} belongs to two adjacent simplices $\tau_1(v^1, \gamma)$ and $\tau_2(w^1, \delta)$ of G_m and*

$$x = v(\tau_1) + \sum_{h \in T} \mu_h q(\gamma_h)/f_{m+1},$$

then for the same μ_h

$$x = w(\tau_2) + \sum_{h \in T} \mu_h q(\delta_h)/f_{m+1}.$$

This important fact will be often used in the replacement steps of the algorithm.

3. The algorithm

Let us assume we want to compute a fixed point of a continuous function g from S^n into itself. Each point $x \in S^n$ is labelled by an integer label $l(x)$ with $l(x) = i$ if $i = \min\{j \mid x_j - g_j(x) \geq x_h - g_h(x)$ for all h, and $x_j > 0\}$. Note that this labelling rule is proper, i.e. $l(x) \neq i$ if $x_i = 0$. A completely labelled n-simplex, i.e. a simplex whose vertices are differently labelled, yields a good approximation of a fixed point. The accuracy of this approximation becomes better according as the grid becomes finer. A vertex (x, f_m) of $S^n \times \{f_m\}$ receives the label $l(x)$. For some given integer $f_1 \geq 2$ the algorithm starts with the simplex ϕ_0 being the convex hull of the vertices of $S^n \times \{f_0\}$ and an arbitrary gridpoint v of $S^n \times \{f_1\}$, e.g. v is the gridpoint nearest to the barycenter of $S^n \times \{f_1\}$. $S^n \times |f_0, f_1|$ is now triangulated as described in the previous section with $\sigma_0 = S^n$ and v being its centrepoint $u(\sigma_0)$. Note that $S^n \times \{f_0\}$ is a completely labelled facet of ϕ_0, *i.e.* all its vertices are differently labelled. The algorithm proceeds now along a path of adjacent simplices with completely labelled common facets starting from ϕ_0. Clearly, since the labelling is proper, the algorithm must find within a finite number of iterations a simplex with a completely labelled facet of $S^n \times \{f_1\}$, say $\sigma_1(u^1, \beta)$. Note that the intersection of the path of adjacent simplices with $S^n \times \{f_1\}$ is the path of adjacent faces generated by Van der Laan and Talman's algorithm [4]. Now, we choose an integer k_1 (≥ 2) and triangulate $S^n \times [f_1, f_2]$ as described in the previous section with σ_1 as the starting simplex and an arbitrary gridpoint $u = \sum \lambda_i^1 u^i / k_1$ of G_2 in σ_1 as its centrepoint $u(\sigma_1)$. The algorithm continues the path of adjacent simplices with completely labelled common facets by computing $l(v(\sigma_1))$. When again a completely labelled simplex of $S^n \times \{f_1\}$ is found, the algorithm proceeds the path in the triangulation of $S^n \times [f_0, f_1]$, until again a completely labelled simplex, say $\tau(v^1, \gamma)$, in $S^n \times \{f_1\}$ is found. Then the algorithm continues as above in $S^n \times [f_1, f_2]$ by computing $l(v(\tau))$, where $v(\tau) = \Sigma \lambda_i^1 v^i / k_1$ is the centrepoint of τ.

Within a finite number of steps a completely labelled simplex of $S^n \times \{f_2\}$ will be found since the number of simplices in $S^n \times [f_0, f_1]$ and $S^n \times [f_1, f_2]$ is finite and replacement steps are unique and feasible, the latter because of the proper labelling. Now again a factor of incrementation can be chosen as well as a centrepoint and the algorithm continues by computing its label etc. The algorithm can be terminated if a fine enough grid is reached. Clearly, within a finite number of iterations the algorithm finds a completely labelled simplex of this grid.

Using vector labelling the algorithm starts from the same simplex ϕ_0 with the system of $n + 1$ linear equations $Iy = e$ by computing $l(v(\sigma_0), f_1)$ where $l(x, f_m) = x - g(x) + e$, $m \geq 1$ and $l(x, f_0) = x$, and where $e = (1, 1, \ldots, 1)$ (see [9]). To compute a fixed point of an mapping ψ, we define for $m \geq 1$, $l(x, f_m) = x - g^m(x) + e$ where g^m is a linear approximation to ψ with respect to G_m. The algorithm proceeds with alternating pivot and replacement steps and converges using analogous arguments.

4. The replacement steps

Let ϕ be a simplex of $S_n \times [f_m, f_{m+1}]$ generated by the algorithm such that there are a simplex $\tau(u^1, \beta)$ of $S^n \times \{f_m\}$ with

$$u(\tau) = \sum_{i=1}^{n+1} \lambda_i^m u^i$$

as its centrepoint, a subset T of t elements of I_{n+1}, a permutation $\pi^T = (\pi_1, \dots, \pi_t)$ of the elements of T, and a nonnegative integer vector $R = (R_1, \dots, R_{n+1})$ with the following conditions:

(1) The intersection of ϕ and G_m is the convex hull of the $n+1-t$ vertices u^i of $\tau(u^1, \beta)$ with $\beta_i \notin T$. This set of vertices u^i is called the set of active vertices of τ, whereas the others are called inactive.

(2) The intersection of ϕ and G_{m+1} is the face $\alpha(v^0, \pi^T)$ with vertices $v^0, \dots, v^t$ in τ such that

$$v^0 = u(\tau) + \sum_{j=1}^{n+1} R_j q(j)/f_{m+1}$$

and

$$v^i = v^{i-1} + q(\pi_i)/f_{m+1}, \quad i = 1, \dots, t.$$

(3) $R_j = 0$ for all $j \notin T$.

It is easy to see that these conditions are fulfilled for the simplex ϕ_m which is the convex hull of $\sigma_m \times \{f_m\}$ and its centrepoint $v(\sigma_m)$ on f_{m+1}. In particular the conditions are satisfied for the simplex with which the algorithm starts, viz. the convex hull of $S^n \times \{f_0\}$ and its centrepoint $v(\sigma_0)$ on f_1. Now any replacement step can be described by adapting the simplex $\tau(u^1, \beta)$ of $S^n \times \{f_m\}$, the subset T, the permutation π^T and the vector R. As described in the previous section two facets of ϕ are completely labelled. So only two vertices have the same label and one of them is the last vertex generated by the algorithm. Then the other must be replaced.

Now two cases can occur,

Case A: the vertex to be replaced is an active vertex of $\tau(u^1, \beta)$ say u^{i_0};

Case B: the vertex to be replaced is a vertex of $\alpha(v^0, \dots, v^t)$, say v^{i_0}.

Case A: u^{i_0} *is the only active vertex of* τ. Then α is a completely labelled simplex of S^n on level f_{m+1}. The simplex $\tau(u^1, \beta)$ is set equal to the completely labelled simplex $\alpha(v^0, \dots, v^{n+1})$, in the way as described in Section 2, R is set equal to zero and T becomes the empty set. The algorithm continues by computing the label of

$$u(\tau)=\sum_{i=1}^{n+1}\lambda_i^{m+1}u^i,$$

the centrepoint of $\tau(u^1,\beta)$, where the λ_i^{m+1}, $i=1,\dots,n+1$ can be chosen arbitrarily if this level is reached for the first time.

u^{i_0} *is not the only active vertex of* τ. Two cases can occur, either $\lambda_{i_0}^m=0$ and $\beta_{i_0-1}\notin T$ or not (if $i_0=1$, then $i_0-1=n+1$). In the latter case $v^{t+1}=v^t+q(\beta_{i_0})/f_{m+1}$ is a gridpoint of G_{m+1} in τ, as follows from the proof of Theorem 2.1. Then T is set equal to $T\cup\{\beta_{i_0}\}$, i.e. u^{i_0} becomes an inactive vertex of τ, π^T becomes $(\pi_1,\dots,\pi_t,\beta_{i_0})$, whereas the simplex τ and the vector R do not change. Now conditions (1)–(3) are satisfied and $l(v^{t+1})$ is computed.

In the other case $v^t+q(\beta_{i_0})/f_{m+1}$ is a gridpoint of G_{m+1} not in τ, and τ is adapted according to Table 1 by replacing u^{i_0}. Then the conditions (1)–(3) are satisfied for the new ϕ with the same R, T and π^T. It follows immediately from Corollary 2.2 that the new simplex ϕ is indeed a simplex of the triangulation adjacent to the previous one. Now the label of the new vertex of τ is computed.

Case B: If $t>0$ three cases can happen:

Case B_1: $j_0=0$,

Case B_2: $1\le j_0\le t-1$,

Case B_3: $j_0=t$.

In Case B_1 either $\beta_{r-1}\notin T$ and $R_{\beta_r}+1=\lambda_r^m$, where r is the index such that $\pi_1=\beta_r$ ($r-1=n+1$ if $r=1$), or not. In the latter case we adapt π^T and R according to Table 2 for $s=0$ and we continue the algorithm by computing the label of the new vertex of α.

In the other case $v^t+q(\pi_1)/f_{m+1}$ is not a grid-point of G_{m+1} in τ. Now π_1 becomes β_{r-1}, R_{β_r} and $R_{\beta_{r-1}}$ are interchanged, T becomes $T\cup\{\beta_{r-1}\}/\{\beta_r\}$, and the simplex $\tau(u^1,\beta)$ is adapted according to Table 1 by replacing the inactive vertex u^r. The algorithm continues by computing the label of the new vertex v^0 of α.

In Case B_2 either $\pi_{j_0}=\beta_{r-1}$ ($r-1=n+1$ if $r=1$) and $R_{\beta_r}-R_{\beta_{r-1}}=\lambda_r^m$, where r is the index such that $\pi_{j_0+1}=\beta_r$, or not. In the latter case π and R are adapted according to Table 2 for $s=j_0$ and the algorithm continues by computing the label of the new vertex of α.

In the other case $v^{j_0-1}+q(\pi_{j_0+1})/f_{m+1}$ is not a gridpoint of G_{m+1} in τ. Now $\tau(u^1,\beta)$ is adapted according to Table 1 by replacing u^r, π^T according to Table 2 for $s=j_0$, and R_{β_r} and $R_{\beta_{r-1}}$ are interchanged. Then the label of the new vertex of α is computed.

In Case B_3 if $R_{\pi_t}\ge 1$, π^T and R are adapted according to Table 2 and $l(v^0)$ is computed. Otherwise, v^t must be replaced by an inactive vertex of $\tau(u^1,\beta)$, viz.

Table 1
i is the index of the vector which must be replaced

	u^1 becomes	β becomes
$i=1$	$u^{n+1}+q(\beta_1)/f_m$	$\beta_{n+1},\beta_2,\dots,\beta_n,\beta_1$
$2\le i\le n+1$	u^1	$\beta_1,\dots,\beta_{i-2},\beta_i,\beta_{i-1},\beta_{i+1},\dots,\beta_{n+1}$

Table 2
s is the index of the vertex which has to be replaced

	π^T becomes	R becomes
$s=0$	$(\pi_2, \dots, \pi_t, \pi_1)$	$R+e(\pi_1)$
$1 \le s \le t-1$	$(\pi_1, \dots, \pi_{s-1}, \pi_{s+1}, \pi_s, \pi_{s+2}, \dots, \pi_t)$	R
$s=t$	$(\pi_t, \pi_1, \dots, \pi_{t-1})$	$R-e(\pi_t)$

the vertex u^r where r is the index with $\beta_r = \pi_t$. Then T becomes $T/\{\pi_t\}$, π^T becomes $(\pi_1, \dots, \pi_{t-1})$ and the algorithm continues by computing $l(u^r)$. From the proof of Theorem 2.1 and Corollary 2.2 it follows that all these replacement steps indeed generate an $(n+1)$-simplex of the triangulation adjacent to the previous simplex.

The only thing to be treated is the case that $t=0$, i.e. $T=\emptyset$, and v^0 has to be removed. Note that v^0 must be the centrepoint of $\tau(u^1, \beta)$ and that τ is a completely labelled subsimplex of $S^n \times \{f_m\}$. Now the algorithm continues with generating simplices of $S^n \times [f_{m-1}, f_m]$. Therefore we have to compute both the simplex $\sigma(w^1, \gamma)$ of $S^n \times \{f_{m-1}\}$ such that all vertices of $\tau(u^1, \beta)$ are gridpoints of G_m in $\sigma(w^1, \gamma)$, and the vertex of $\sigma(w^1, \gamma)$, say w^{i_0}, which is connected with all vertices of τ. To do so, we choose arbitrarily an interior point of $\tau(u^1, \beta)$, say x, and we calculate for $h = m-1$, $\alpha_i(x)$, $i=1, \dots, n$ as described in Section 2.

Let $\bar{\alpha}_i$ be the entier of $\alpha_i(x)$, $i=1, \dots, n$ and let $\bar{w}^1$ be the grid point of S^n such that

$$\bar{w}^1 = e(1) + \sum_{i=1}^{n} \bar{\alpha}_i q(i)/f_{m-1}.$$

Since x is an interior point of $\tau(u^1, \beta)$, $\alpha_i(x) - \bar{\alpha}_i$ is different for all i. Now let $\bar{\gamma}$ be the permutation of the elements of I_{n+1} such that $\alpha_{\gamma_1}(x) - \bar{\alpha}_{\gamma_1} > \bar{\alpha}_{\gamma_2}(x) - \bar{\alpha}_{\gamma_2} > \dots > \alpha_{\gamma_n}(x) - \bar{\alpha}_{\gamma_n}$ and $\gamma_{n+1} = n+1$. Then $\tau(\bar{w}^1, \bar{\gamma})$ is the simplex on level f_{m-1} containing x. Hence, $\sigma(w^1, \gamma)$ is the simplex $\tau(\bar{w}^1, \bar{\gamma})$ represented in the way described in Section 2. It is left to find w^{i_0}. It can easily be seen that

$$u^1 = w(\sigma) + \sum_{j=1}^{n} \theta_j q(j)/f_m \quad \text{with } \theta_j = \sum_{i=1}^{j} (w_i(\sigma) - u_i^1),$$

where $w(\sigma)$ is the centrepoint of $\sigma(w^1, \gamma)$. Define $z = \min_{j=1,\dots,n+1} \theta_j$, where θ_{n+1} is set equal to zero, and let $\bar{\theta}_j = \theta_j - z$. Since $\sum_{i=1}^{n+1} q(i) = 0$, we obtain

$$u^1 = w(\sigma) + \sum_{j=1}^{n+1} \bar{\theta}_j q(j)/f_m.$$

From

$$u^i = u^1 + \sum_{j=1}^{i-1} q(\beta_j)/f_m$$

it follows that

$$u^i = w(\sigma) + \sum_{j=1}^{n+1} \theta_j^i q(j)/f_m, \quad i = 1, \dots, n+1,$$

where $\theta_j^i = \bar{\theta}_j + 1$ if there exists an index $h \in \{1, \dots, i-1\}$ such that $\beta_h = j$, and where $\theta_j^i = \bar{\theta}_j$ if not. Let $H(u^i)$ be the set $\{j \mid \theta_j^i = 0\}$. Clearly $|H(u^1)| \geq 1$, $|H(u^{n+1})| \leq 1$ and $|H(u^i)| - |H(u^{i+1})| = 0$ or 1 for $i = 1, \dots, n$. So there exists an index j such that $|H(u^j)| = 1$. Let $H(u^j) = \{\beta_{j_0}\}$. Then w^{i_0} is the desired vertex of $\sigma(w^1, \gamma)$, where i_0 is the index such that $\beta_{j_0} = \gamma_{i_0}$. The conditions (1)–(3) are now again satisfied for $T = I_{n+1} \setminus \{\beta_{j_0}\}$, $\pi^T = (\beta_{j_0+1}, \dots, \beta_{n+1}, \beta_1, \dots, \beta_{j_0-1})$, $R_i = \theta_i^{j_0+1} - \min_s \theta_s^{j_0+1}$, for $i = 1, \dots, n+1$, and $\tau(u^1, \beta)$ is set equal to $\sigma(w^1, \gamma)$. The algorithm then continues by computing $l(u^{i_0})$.

All these cases together give a formal description of the replacement steps. Because of the proper labelling, all replacement steps are feasible.

5. The application on $R^n \times [0, \infty)$

To triangulate $R^n \times [0, \infty)$ in an appropriate way, we choose a sequence $f_0, f_1, \dots,$ such that f_{m+1} is a multiple k_m of f_m, $m \geq 1$, $f_1 > 0$, and $f_0 = 0$. On level f_m, R^n is triangulated with grid size f_m, $m \geq 1$, i.e. the simplices on level f_m are $\tau(v^1, \gamma)$ with vertices $v^1, \dots, v^{n+1}$, where γ is a permutation of the elements of I_{n+1}, such that $v^1 = \sum_{i=1}^n \alpha_i p(i)/f_m$, where α_i is an integer, $i = 1, \dots, n$, $v^{j+1} = v^j + p(\gamma_j)/f_m$, $j = 1, \dots, n$ and such that $v^1 = v^{n+1} + p(\gamma_{n+1})/f_m$, with $p(i)$ the ith column of an $n \times (n+1)$ triangulation matrix P. For instance, if $p(i) = e(i)$, $i = 1, \dots, n$, $p(n+1) = -e$, the K-triangulation is obtained, and if $p(i, i) = n + \sqrt{n+1}$, $p(i, n+1) = -(1 + \sqrt{n+1})$, $i = 1, \dots, n+1$ and $p(i, j) = -1$ otherwise, we have the optimal triangulation of R^n according to the SC measure within the class of the so-called (α, β) triangulations (see Van der Laan and Talman [7]). Note that the zero-point is a grid point. Again, as on S^n, each simplex on level f_m has $n+1$ representations. The procedure of Section 2 then represents each simplex in a unique way if for a given gridpoint y the n-vector $\alpha(y)$ is defined by $\alpha(y) = f_m \tilde{P}^{-1} y$ where $\tilde{P}$ is the $n \times n$-matrix consisting of the first n columns of P.

It is sufficient to describe only the triangulation of $R^n \times [f_m, f_{m+1}]$ for some $m \geq 1$ and that of $R^n \times [0, f_1]$. Combining then these triangulations for all pairs $[f_i, f_{i+1}]$, $i \geq 0$, we obtain immediately the triangulation of $R^n \times [0, \infty)$. The triangulation of $R^n \times [f_m, f_{m+1}]$ for some $m \geq 1$ is done in the same way as described in Section 2 for $S^n \times [f_m, f_{m+1}]$. So, let $\sigma_m(u^1, \beta)$ be the starting simplex on level f_m, and let $u(\sigma_m) = \sum_{i=1}^{n+1} \lambda_i^m u^i / k_m$ be its centrepoint for arbitrarily chosen $\lambda_1^m, \dots, \lambda_{n+1}^m$ such that $\sum_{i=1}^{n+1} \lambda_i^m = k_m$. To subdivide each simplex $\tau(v^1, \gamma) \times [f_m, f_{m+1}]$ represented in the correct way, we define again the regions $A^\tau(T)$, with T a proper subset of I_{n+1}, and connect all grid points x on level f_{m+1} in region $A^\tau(T)$ with the vertices v^i of τ on level f_m for $i \notin T$. Combining the triangulations of $\tau \times [f_m, f_{m+1}]$ over all simplices τ, we get a consistent triangulation of $R^n \times [f_m, f_{m+1}]$.

Finally, we have to triangulate $R^n \times [0, f_1]$. Let v be an arbitrarily chosen gridpoint on level f_1. Let $\sigma_0(u^1, \dots, u^{n+1})$ be the simplex with vertices $u^i = v - p(i)$, $i = 1, \dots, n+1$. Instead of triangulating $R^n \times [0, f_1]$, we will actually triangulate $(\sigma_0 \times \{0\}) \cup (R^n \times (0, f_1])$. We define the set of grid points $A^0(T)$ on level f_1 for any $T \subset I_{n+1}$, by

$$A^0(T) = \{x \in R^n \mid x = v + \textstyle\sum_{h\in T} \mu_h q(h)/f_1 \text{ for positive integers } \mu_h,\ h \in T\}.$$

Observe that these regions partition all gridpoints on level f_1, whereas the above defined regions $A^\tau(T)$ partition only the gridpoints in τ. By connecting all gridpoints in $A^0(T)$ on level f_1 with the vertices u^i of σ_0 on level zero for $i \notin T$, $(\sigma_0 \times \{0\}) \cup (R^n \times (0, f_1])$ is triangulated. Let us assume we want to compute a fixed point of a function f from R^n to R^n. The $n+1$ vertices of σ^0 are artificially labelled by $l(u^i) = e(i)$ in case of vector labelling, and by $l(u^i) = i$ in case of integer labelling, i.e. σ_0 is a completely labelled facet. A point x on level f_m is labelled by the $n+1$ vector $l(x)$ where $l_i(x) = f_i(x) - x_i + 1$, $i = 1, \dots, n$, and $l_{n+1}(x) = 1$, in case of vector labelling. In case of integer labelling $l(x) = i$ with $i = \min\{j \mid g_j(x) - x_j \geq g_h(x) - x_h \text{ for all } h\}$ if $g_i(x) - x_i \geq 0$, and $l(x) = n+1$ otherwise. If we have a mapping ψ, g is a linear approximation to ψ. In that case integer labelling is not appropriate. Conditions to guarantee the existence of a fixed point are given in [9, 10, 14]. The algorithm now starts with the simplex being the convex hull of σ_0 on level f_0 and the point v on level f_1 by computing $l(v)$, and proceeds along a path of adjacent simplices with completely labelled common facets. The intersection of the path of simplices of $(\sigma_0 \times \{0\}) \cup (R^n \times (0, f_1])$ with $R^n \times \{f_1\}$ is exactly the path of simplices generated by the algorithm described by Van der Laan and Talman [6]. The above used interpretation of their algorithm with $n+1$ artificially labelled points can be found in Todd [15]. To implement the algorithm on the computer the artificially labelled points can be deleted as described in [6]. As soon as a completely labelled simplex on level f_1 is found, this simplex will be the starting simplex σ_1 on level f_1. The k_1 and λ_i^1's are fixed and the algorithm continues with simplices in $R^n \times [f_1, \infty)$. All replacement steps in $R^n \times [f_1, \infty)$ are the same as those for $S^n \times [1, \infty)$.

It is left to consider the case that again a completely labelled simplex, say $\tau(v^1, \gamma)$, on level f_1 is found, i.e. that the algorithm must proceed with simplices in $(\sigma_0 \times \{0\}) \cup (R^n \times (0, f_1])$. Formally, we have to determine the unique vertex of σ_0 which is connected with all vertices of τ. If u^{i_0} is that vertex, determined analogously as in Section 4, T becomes $I_{n+1} \setminus \{i_0\}$, π^T becomes $(\gamma_{j_0+1}, \dots, \gamma_{n+1}, \gamma_1, \dots, \gamma_{j_0-1})$ where $\gamma_{j_0} = i_0$, and R becomes as described at the end of Section 4. Of course the simplex on level f_0 becomes σ_0. Equivalently the algorithm is continued as described in [6], until again a completely labelled simplex on level f_1 is found.

6. Conclusions

In the algorithm of Eaves [1] and Eaves and Saigal [2] the factor of incrementation must always be equal to two. So, their algorithm can not be used to obtain quadratic convergence. Therefore, Saigal [12] and Saigal and Todd [13] developed an acceleration technique, using a restart method in a homotopy algorithm, to achieve quadratic convergence when the underlying function is continuously differentiable and its derivative satisfies a Lipschitz condition. In the homotopy method using the triangulation introduced in this paper any factor of incrementation between two successive levels can be chosen, which allows to obtain quadratic convergence without the necessity of making restarts.

Moreover, in our method a starting point on a new level can be chosen on the ground of information obtained earlier. For integer labelling one could choose the gridpoint nearest (lexicographically) to the barycenter of the completely labelled simplex on the current level and in case of vector labelling a gridpoint close to the approximation of the fixed point. Concerning the grid size and the centrepoint on levels visited earlier it is sufficient to keep in storage only the numbers λ_i^m, $i = 1, \dots, n+1$. The numbers determine the grid size f_{m+1} as well as the starting point $u(\sigma_m)$ on level f_{m+1} completely. Computational experience must show the usefulness of the technique developed in this paper. Two issues for study are to decide how information can be best used and secondly the dependence on the underlying triangulation of S^n or R^n. In T^n, the affine hull of S^n, the triangulation introduced by Van der Laan and Talman [7] seems to be a good one, which can also be implemented on S^n by projecting points outside S^n on the boundary.

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